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Nonextensive statistics and incomplete information

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Abstract. We comment on some open questions and theoretical peculiarities in Tsallis nonextensive statistical mechanics. It is shown that the theoretical basis of the successful Tsallis' generalized exponential distribution shows some worrying properties with the conventional normalization and the escort probability. These theoretical difficulties may be avoided by introducing an so called incomplete normalization allowing to deduce Tsallis' generalized distribution in a more convincing and consistent way.

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1 Introduction

It is well known that Boltzmann-Gibbs statistics (BGS) is inadequate for treating some complex systems. These are systems with complex or long term interactions and correlations, systems showing often distribution laws different from the usual ones (Gauss, Poisson), systems in chaotic or fractal states and often related to nonextensive phenomena in energy, entropy, heat, and other quantities. Some examples of the failures of BGS are given in reference [1]. We see that we need a new statistical mechanics fundamentally different from BGS. We can even conjecture that a new kind of statistical theory may be necessary for complex random phenomena because the validity of the actual statistical method is subject to some ideal conditions [2]. A brief discussion on this topic will be given in the present paper.

In 1988, in a historical paper [3,4], Tsallis founded a nonextensive statistical mechanics which, in its most recent version, gives following canonical distribution functions:

$$p_i = \frac{\left[1-(1-q)\frac{\beta}{\sum_i^w p_i^q}(e_i-U)\right]_\dagger^{\frac{1}{1-q}}}{Z} \qquad (1)$$

with

$$Z = \sum_i^w \left[1-(1-q)\frac{\beta}{\sum_i^w p_i^q}(e_i-U)\right]_\dagger^{\frac{1}{1-q}} \qquad (2)$$

where $[y]_\dagger = y$ if $y \geq 0$ and $[y]_\dagger = 0$ otherwise (Tsallis cut-off). i is a state point in phase space, w the number of all accessible phase space points for the considered system, e_i the energy of the system at the state i, β the Lagrange multiplier of the constraint on the internal energy U given by

$$U = \frac{\sum_i^w p_i^q e_i}{\sum_i^w p_i^q} = \frac{\sum_i^w p_i^q e_i}{Z^{1-q}} \qquad (3)$$

which is used by Tsallis to overcome some theoretical difficulties [4,5]. Equations (1, 2) mean

$$\sum_i^w p_i = 1, \qquad (4)$$

which is logical if we consider the fact that w represents all possible states of the system. It should be noticed that, in the limit $q = 1$, equation (1) becomes BGS, *i.e.* $p_i = \frac{1}{Z}\exp(-\beta e_i)$.

We refer to the generalized exponential distribution equation (1) as Tsallis distribution function (TDF) which is proved to be indeed very useful and efficacious in describing some systems with complex or long term interactions. Many successful and convincing applications was published over the last 10 years concerning different systems showing peculiar distribution laws (the reader is referred to references [1,6,7] for updated comments on this subject).

In spite of the successes of TDF, researchers continue to work on its theoretical foundation, because there are still various open questions. One notes peculiar theoretical properties which sometimes are not very easy to understand and deserve to be investigated. In this paper, we would like to discuss some fundamental aspects of the last version of Tsallis theory and present some observations. We also show a possible solution to the problems with a modifications in the theoretical basis of nonextensive statistics.

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2 Brief history of TDF

Let us begin by a brief review of the historical path of TDF.

In 1988, Tsallis proposed the following generalized statistical entropy:

$$S = -k\frac{1-\sum_{i=1}^{w} p_i^q}{1-q}, (q \in R) \tag{5}$$

which he maximized according to Jaynes principle with two constraints: equation (4) and the mathematically suitable expectation value of energy U:

$$U = \sum_{i}^{w} p_i e_i. \tag{6}$$

This definition is completely legitimate if p_i is considered as probability. This approach led to the first version of TDF, *i.e.*

$$p_i = \frac{[1-(q-1)\beta e_i]_{\dagger}^{\frac{1}{q-1}}}{Z} \tag{7}$$

with

$$Z = \sum_{i}^{w} [1-(q-1)\beta e_i]_{\dagger}^{\frac{1}{q-1}}. \tag{8}$$

Later, it was found that Tsallis-1 had some shortcomings such as the thermodynamic instability of entropy and the incapacity to deduce some power-law distributions [4,8–11]. But quite before the first published criticisms [9,11] on this theory, another version of TDF was proposed by Curado and Tsallis [12]. This second version, say, Curado-Tsallis, replaced equation (6) by:

$$U = \sum_{i}^{w} p_i^q e_i. \tag{9}$$

Curado-Tsallis formalism allows to get a mathematical elegance with the conventional Legendre transformation of thermodynamics:

$$\frac{\partial S}{\partial U} = \frac{1}{T} = k\beta \tag{10}$$

where T is the absolute temperature. This elegance is lost by Tsallis-1 version which gives:

$$\frac{\partial S}{\partial U} = \frac{1}{T} = Z^{1-q}k\beta \tag{11}$$

where U is no longer a simple function dependent only on Z just like in the BGS case[1].

[1] In Tsallis-1 version, we have $U = -\frac{1}{qZ}\frac{\partial}{\partial\beta}\sum_i[1-(q-1)\beta e_i]_{\dagger}^{\frac{q}{q-1}}$, which nevertheless tends to BGS relation $U = -\frac{\partial}{\partial\beta}\ln Z$ when $q \to 1$.

The Curado-Tsallis formalism once again gives equation (7) with a change $(q-1) \to (1-q)$, and is successfully applied to systems showing non Gaussian distribution laws as soon as its proposal. Nevertheless, the anomalous relation between the normalization equation (4) and the averages equation (9) remains an open question. Rigorously speaking, supposing equation (4), we logically write equation (6) for an observable average. In other words, equation (9) means that p_i^q is the observable probability but is not normalized and that p_i is an imaginary probability which is normalized but never used in practice, *i.e.* p_i is not observable if equation (4) holds.

This second formalism, according to some authors [4,5,11], shows some fundamental difficulties. For example: 1) the average of a constant is not constant; 2) the total energy of two systems without any interaction is not the sum of the energies of each system; 3) the zeroth law does not hold; 4) the invariance of probability with uniform energy translation is missing.

The third or last version of TDF is that mentioned at the beginning of this paper. It replaces the unnormalized average by a normalized one equation (3) and so resolved the above mentioned problems.

When writing this manuscript, we saw another proposal by Martinez *et al.* [13] concerning the energy constraint for entropy maximization. They propose using

$$\sum_{i}^{w} p_i^q (e_i - U), \tag{12}$$

as the constraint but with U given by equation (3). This approach overcomes a mathematical difficulty of the third version of TDF, that is, the auxiliary function $A(p)$ for entropy maximization:

$$A = -\frac{1-\sum_{i=1}^{w} p_i^q}{1-q} - \alpha\left(\sum_{i}^{w} p_i - 1\right) - \beta\left(\frac{\sum_i^w p_i^q e_i}{\sum_i^w p_i^q} - U\right) \tag{13}$$

does not necessarily have a maximum. Because, first, $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i \mathrm{d}p_j}$ is not diagonal, *i.e.*, $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i \mathrm{d}p_j} \neq 0$ when $i \neq j$. Second, even if we calculate only $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2}$, we get:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2} = -qp_i\left[Z^{1-q} - q\beta p_i^{2q-1} Z^{2q-2}(e_i - U)\right]. \tag{14}$$

To ensure $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2} < 0$ for a maximum entropy, we have to put $[Z^{1-q} - q\beta p_i^{2q-1} Z^{2q-2}(e_i - U)] > 0$, which, evidently, can not be always valid. Martinez *et al.* overcame this difficulty with equation (12) leading to $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i \mathrm{d}p_j} = 0$ for $i \neq j$ and $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2} = -qp_i Z^{1-q} < 0$. The entropy maximum is ensured.

This maximization still leads to equation (1) as probability distribution. In fact, this proposal is a composition of the Curado-Tsallis maximization with $\sum_i^w p_i^q e_i$ and another constraint $\sum_i^w p_i^q U$. The latter can be called invariance constraint since its only role is to ensure the invariance of the resulted distribution with respect to uniform translation of the energy levels e_i. Due to the same distribution functions, this scenario has the same essential characteristics as the last version of TDF, about which we will discuss some open questions in the following section.

3 Some questions about TDF

1. Let us begin by presenting an observation about the maximum entropy principle of Jaynes. This principle asserts that, in order to obtain the correct probability distribution, it suffices to introduce into the entropy maximization physical conditions or constraints related to observable quantities, that is, the normalization of the probability, the expectation, variance or higher moments. Supposed equation (4) as normalization, the only observable expectation value or higher moments must be defined with equation (6). The introduction of equation (9) or equation (3) does not conform with this principle, because we do not know whether these averages really represent observable quantities or not. One may say that the successful applications of these formalisms confirm the observability of these averages. But in this case, p_i becomes non observable in turn. So the normalization equation (4) should disappear in the entropy maximization. This observability question deserves to be carefully studied. Indeed, if necessary, we could reject any old principle and introduce new ones under the condition that the resulted theory be useful and self-consistent.
2. The second question was discussed by Raggio [14]. If the expectation equation (3) satisfies the constraint associated with the linearity in the observables, *i.e.*

$$\overline{x+y} = \overline{x} + \overline{y} \tag{15}$$

for two independent observables x̂ and ŷ, it does not satisfy the linearity in the state (or distribution), *i.e.*

$$\overline{x}[\lambda p(1) + (1-\lambda)p(2)] = \lambda \overline{x}[p(1)] + (1-\lambda)\overline{x}[p(2)] \tag{16}$$

where $p(1)$ and $p(2)$ are two normalized probability distributions, nor does equation (9) which violates even equation (15). The origin of these violations is logically the *discrepancy between the average equation (3) (or (9)) and the normalization equation (4)* according to which the joint probability is $p(x+y) = p(x)p(y)$ and the probability summation is $p(1+2) = p(1)+p(2)$. If the average is calculated with p^q instead of p, the problem is evident since p^q is not normalized and $p^q(1+2) \neq p^q(1) + p^q(2)$. Indeed,

$$\overline{x}[\lambda p(1) + (1-\lambda)p(2)] = \frac{\sum_i [\lambda p_i(1) + (1-\lambda)p_i(2)]^q x_i}{\sum_i [\lambda p_i(1) + (1-\lambda)p_i(2)]^q} \neq \lambda \frac{\sum_i p_i^q(1) x_i}{\sum_i p_i^q(1)} + (1-\lambda)\frac{\sum_i p_i^q(2) x_i}{\sum_i p_i^q(2)} \cdot \tag{17}$$

Equations (15, 16) are to be satisfied for that x̂, ŷ, $p(1)$ and $p(2)$ be observable and have physical signification. This problem naturally leads us to the following question: does the observability of the normalization equation (4) is incompatible with that of the expectation value given by equation (3) or (9)?
3. The third question we would like to discuss is about the absence of analytic energy correlation. We know that the correlations in entropy can be calculated *a priori* with mathematical rigor. But the correlation in any other observable quantities can not be calculated with the average equation (3).
Let us suppose an isolated system C composed of two subsystems A and B of which the distributions satisfy

$$p_{ij}(C) = p_i(A)p_j(B). \tag{18}$$

In nonextensive statistics, this hypothesis of multiplication law means that A and B are correlated and gives the correlation term associated with energy. From equations (1, 2) and (3), we straightforwardly obtain:

$$e_{ij}(A+B) - U(A+B) = [e_i(A) - U(A)]\sum_i p_i^q(B) + [e_j(B) - U(B)]\sum_j p_j^q(A) + (q-1)\beta[e_i(A) - U(A)][e_j(B) - U(B)]. \tag{19}$$

Without additional hypothesis, this equality does not lead to any explicit relation between the Hamiltonians $H(A+B)$ and $H(A)$ or $H(B)$ neither for the micro-state values e_i nor for the average U. Tsallis and coworkers [4,15] proposed neglecting the correlation term of any observable between the subsystems. So one can write:

$$e_{ij}(A+B) = e_i(A) + e_j(B) \tag{20}$$

and

$$U(A+B) = U(A) + U(B). \tag{21}$$

On the basis of this *extensive energy approximation*, the zeroth law of thermodynamics is claimed to be established for the foundation of nonextensive thermodynamics [15,16]. This problem will be discussed below.

Our question about equations (20, 21) is: if in general the correlation (nonextensive) terms of whatever observable or interactions can be neglected, what is the origin of the nonextensivity of entropy? We know that entropy should be a continuous function of the distributions which in turn are continuous functions of the observables.

In addition, allowing equations (20, 21), we lose the equalities equation (19) and equation (18), which are crucial for the nonextensive theory. If equation (18) fails, we cannot in fact find even the entropy correlation given by

$$S(A+B) = S(A) + S(B) + \frac{1-q}{k}S(A)S(B). \tag{22}$$

4. Zeroth law of thermodynamics can not be established without neglecting the correlation energy. That is to say that the Lagrange multiplier β is no longer a "meter" measuring the thermodynamic equilibrium. We will show why.

   We take once again the above mentioned isolated system C composed of two subsystems A and B in equilibrium. From equation (22), we get, for a small variation of the total entropy:

$$\begin{aligned}\delta S(A+B) &= \left[1+\frac{1-q}{k}S(B)\right]\delta S(A) \\ &+ \left[1+\frac{1-q}{k}S(A)\right]\delta S(B) \\ &= \left[1+\frac{1-q}{k}S(B)\right]\frac{\partial S(A)}{\partial U(A)}\delta U(A) \\ &+ \left[1+\frac{1-q}{k}S(A)\right]\frac{\partial S(B)}{\partial U(B)}\delta U(B).\end{aligned} \tag{23}$$

   Because $\delta S(A+B)=0$, we get

$$\left[1+\frac{1-q}{k}S(B)\right]\frac{\partial S(A)}{\partial U(A)}\delta U(A) + \left[1+\frac{1-q}{k}S(A)\right]\frac{\partial S(B)}{\partial U(B)}\delta U(B) = 0. \tag{24}$$

   Now, in order to proceed, we need the relation between $U(A+B)$ and $U(A)$ or $U(B)$ but it does not exist. As mentioned above, $U(A+B)$ can not be expressed in $U(A)$ and $U(B)$. So no relation can be found between $\delta U(A)$ and $\delta U(B)$ and, as a consequence, the derivative $\frac{\partial S}{\partial U}$ can not be calculated from $\delta S(A+B)=0$. So supposing $U(A+B)=U(A)+U(B)$ or $\delta U(A)=-\delta U(B)$ proposed by Abe and Martinez *et al.* [15,16], we obtain

$$\left[1+\frac{1-q}{k}S(B)\right]\frac{\partial S(A)}{\partial U(A)} = \left[1+\frac{1-q}{k}S(A)\right]\frac{\partial S(B)}{\partial U(B)} \tag{25}$$

   or

$$Z^{q-1}(A)\beta(A) = Z^{q-1}(B)\beta(B) \tag{26}$$

   which is the generalized zeroth law with $Z^{q-1}\beta$ instead of $\beta = 1/kT$ as the measure of the equilibrium. Due to this approximate zeroth law, the implicit distribution function equation (1) becomes explicit for systems in thermal equilibrium since it can be recast as

$$p_i = \frac{[1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{1}{1-q}}}{Z} \tag{27}$$

   where $\lambda = \beta Z^{q-1}$ is now an independent thermodynamic variable. So the theory is reconciled with the old notion of thermal equilibrium to the detriment of the correlation between the components of the system. In our opinion, two important things are lost in this treatment: a) the generality of the nonextensive theory which should be able to tackle correlated problems in taking into account the interactions; b) the generality of the zeroth law which should hold within a theory without any condition, or all thermodynamic laws will become approximate. As a matter of fact, as mentioned above and argued by Guerberoff *et al.* [11], with additive energy or non interacting subsystems, equation (18) and equation (22) do not hold so that the zeroth law equation (26) can not be established. Very recently, a new point of view about thermal equilibrium and nonextensivity shows that equation (18) or equation (22) is required by the existence of thermal equilibrium with Tsallis entropy and should be regarded as a basic assumption of the statistics for equilibrium nonextensive systems. So that appropriate energy nonextensivity satisfying equation (18) is absolutely necessary for the validity of zeroth law within Tsallis statistics [17,18]. We will come back to this issue later in this paper.

5. Now we discuss a mathematical problem. From equation (1), we can write:

$$\begin{aligned}Z &= \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{1}{1-q}} \\ &= \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}+1} \\ &= Z^q \sum_i^w p_i^q [1-(1-q)\lambda(e_i-U)]_\dagger\end{aligned} \tag{28}$$

   Considering the average defined in equations (3, 28) can be recast as:

$$Z = \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}} \tag{29}$$

   or

$$\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{1}{1-q}} = \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}}. \tag{30}$$

   This equality is a basic relation of the theory and must hold for arbitrary value of q, w, β and e_i. Now let us apply it to calculate the inverse temperature β.
   From equation (29), we can write

$$\sum_i^w p_i^q = Z^{1-q} \tag{31}$$

   and

$$S = k\frac{Z^{1-q}(U)-1}{1-q}, \tag{32}$$

Then we calculate the following derivative:

$$\frac{\mathrm{d}S}{\mathrm{d}U} = \frac{k}{Z^q}\frac{\mathrm{d}Z}{\mathrm{d}U}\cdot \tag{33}$$

First we take the Z given by equation (2), we obtain:

$$\frac{\mathrm{d}S}{\mathrm{d}U} = k\lambda Z^{1-q} = k\beta. \tag{34}$$

But if we take the Z of equation (29), we obtain

$$\frac{\mathrm{d}S}{\mathrm{d}U} = \frac{qk\beta}{Z}\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}} \tag{35}$$

which means

$$Z = q\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}}. \tag{36}$$

If we put equation (36) into equation (33) and continue in this way for n times, we will find

$$\begin{aligned} Z &= \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}} \\ &= q\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}} \\ &= q(2q-1)\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{3q-2}{1-q}} \\ &= q(2q-1)(3q-2)\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{4q-3}{1-q}} \\ &= q(2q-1)(3q-2)...(nq-n+1) \\ &\quad\times \sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{(n+1)q-n}{1-q}} \end{aligned} \tag{37}$$

with $n = 0, 1, 2....$ We create in this way a series of equalities which seem not to hold. For example, if we take the second equality of equation (37) and let $q \to 0$, the right-hand side will tend to zero and the left-hand side to $\sum_i^w 1 = w$. The result is $w \to 0$. This same result can also be obtained for $q \to \frac{1}{2}$ if we take the third equality of equation (37) and for $q \to \frac{2}{3}$ with the forth equality and so on. These singular points in q value do not conform with the hypothesis that equation (30) is a general relation of the theory. It seems to us that these equalities are valid only when $q \to 1$ and Z becomes the BGS partition function.

We can also study the equality equation (30) in another way. Let us suppose that $\hat{x}$ is a continuous variable within $0 < x < \infty$. So Z sometimes can be given by

$$Z = \int_0^\infty [1-(1-q)\lambda(x-U)]_\dagger^{\frac{q}{1-q}}\mathrm{d}x. \tag{38}$$

or

$$Z = \int_0^\infty [1-(1-q)\lambda(x-U)]_\dagger^{\frac{1}{1-q}}\mathrm{d}x. \tag{39}$$

In this case, we should put $q > 1$ for Z to be calculated when x is large. The integration of equation (38) is always finite. But equation (39) needs $q < 2$ to be finite. If $q > 2$, the Z of equation (38) can be calculated while that of equation (39) diverges. This paradox naturally disappears for $q \to 1$.

6. Another problem concerning the relation between the average U and the micro-state value e_i or Z arises due to equation (30). Usually, the $U - Z$ relation ($U = -\frac{\partial}{\partial\beta}\ln Z$ in BGS) can be found by introducing the distribution function equation (1) into the average calculus equation (3):

$$\begin{aligned} U &= \frac{\sum_i p_i^q e_i}{Z^{1-q}} \\ &= \frac{1}{Z}\sum_i e_i[1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}} \\ &= -\frac{1}{Z}\left\{\frac{\partial}{\partial\lambda}\sum_i [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{1}{1-q}}\right. \\ &\qquad \left. -\sum_i U[1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{q}{1-q}}\right\} \\ &= -\frac{1}{Z}\left\{\frac{\partial Z}{\partial\lambda} - UZ\right\} \\ &= -\frac{1}{Z}\frac{\partial Z}{\partial\lambda} + U \end{aligned} \tag{40}$$

which leads to, instead of the expected $U - Z$ relation,

$$\frac{\partial Z}{\partial\lambda} = 0. \tag{41}$$

as well as

$$\frac{\partial S}{\partial\lambda} = \frac{\partial}{\partial\lambda}k\frac{Z^{1-q}-1}{1-q} = k\frac{1}{Z^q}\frac{\partial Z}{\partial\lambda} = 0. \tag{42}$$

So the micro-macro relation is impossible to be found if no mechanical quantity can be calculated from its microstate values.

In addition, equation (41) gives rise to another problem similar to that discussed in the precedent part. Equation (41) can be easily verified if we take the standard Z given by equation (2). But If we take the Z in equation (29), we get the following relation

$$\sum_i^w (e_i-U)[1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}} = 0 \tag{43}$$

or

$$\begin{aligned} U &= \frac{\sum_i^w e_i[1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}}}{\sum_i^w [1-(1-q)\lambda(e_i-U)]_\dagger^{\frac{2q-1}{1-q}}} \\ &= \frac{\sum_i^w e_i p_i^{2q-1}}{\sum_i^w p_i^{2q-1}}\cdot \end{aligned} \tag{44}$$

If we repeat the same reasoning with the Z of equation (36), we get

$$U = \frac{\sum_i^w e_i p_i^{3q-2}}{\sum_i^w p_i^{3q-2}}. \quad (45)$$

We can continue in this way with equation (40) and obtain:

$$\begin{aligned} U &= \frac{\sum_i^w e_i p_i^q}{\sum_i^w p_i^q} \\ &= \frac{\sum_i^w e_i p_i^{2q-1}}{\sum_i^w p_i^{2q-1}} \\ &= \frac{\sum_i^w e_i p_i^{3q-2}}{\sum_i^w p_i^{3q-2}} \\ &= \frac{\sum_i^w e_i p_i^{4q-3}}{\sum_i^w p_i^{4q-3}} \\ ... &= \frac{\sum_i^w e_i p_i^{nq-n+1}}{\sum_i^w p_i^{nq-n+1}}. \end{aligned} \quad (46)$$

Which means $\sum_i^w (e_i - U) = 0$ or $U = \sum_i^w e_i/w$ if $q = \frac{n-1}{n}$ with $n = 1, 2, 3....$ *i.e.*, we are led to the microcanonical case. On the other hand, if we calculate U from its definition equation (3) for $q = 1/2$, we will be led to

$$U = \frac{1}{Z}\left(\sum_i^w e_i - \frac{\lambda}{2}\sum_i^w e_i^2 + \frac{\lambda}{2} U \sum_i^w e_i\right) \quad (47)$$

or

$$U = \left(\sum_i^w e_i - \frac{\lambda}{2}\sum_i^w e_i^2\right) \bigg/ \left(Z - \frac{\lambda \sum_i^w e_i}{2}\right), \quad (48)$$

which does not seem to be a microcanonical case.

These mathematical problems discussed above seem to be related directly to the fact that the TDF equation (1) depends only on the difference $e_i - U$. So equation (1) can be regarded as the relative probability distribution with respect to the average energy of the system, but not the real probability with respect to a suitable zero-energy we must choose due to the energy translation variance of TDF. We will come back to this problem at the end of the present paper. We also wonder if the q-independent results for generalized ideal gas [13] have something to do with this relative probability combined with extensive energy approximation for "non interacting particles".

We have discussed some questionable points in the last version of TDF. Some other discussions concerning the relation between TDF of the third version and the law of large number are given in reference [19]. Of course, these are questions that we have to study carefully. For the moment, we do not see how to get out of these theoretical difficulties if we stay in the formalism with the conventional normalization and the expectation value equation (3). In what follows, we will show a possible way out. The main idea is to introduce into Tsallis theory the hypothesis of incompleteness of our knowledge (information) about nonextensive systems.

4 Some considerations concerning statistics

Very recently, a possible alternative theoretical basis for TDF was proposed [2,20]. The new formalism is based on a reflection about the conditions of physical application of the standard probability theory which is sometimes referred to as Kolmogorov probability theory [21]. This reflection leads to a mathematically simpler and coherent nonextensive framework being capable of avoiding the problems discussed above.

This new theoretical framework is referred to as *incomplete statistics* (IS). The basic idea of IS is that *our knowledge about the states (their position and total number in the phase space) of a system may be incomplete and non exact.* This is true at least for complex systems with unknown space-time correlations which can not be exactly described with analytic methods. So the equation of motion must be incomplete in the sense that some interactions are missing and its solution can not yield complete knowledge about the systems. On the other hand, Kolmogorov probability theory is founded on the hypothesis that we know all the possible states of the studied system or that the maximal information is complete and can provide definite answers for all questions that can be asked of the system. When we carry out a summation or an integration of probability in phase-space, we suppose that this is done over the possible states which can be determined by the equation of motion. This assumption is logical if and only if all interactions and space-time correlations are well-known or their unknown parts are negligible, as in the case of BGS or of other successful probabilistic sciences.

5 Incomplete normalization

If the incompleteness of our knowledge is not negligible, it is no longer sure that we sum over the possible states of the system simply because we do not know all of them. What we can do is to take the known states or events suggested by the equation of motion or by our knowledge. Their number, say v, may be greater or smaller than w, the real number of all possible states. As a consequence, the normalization condition is reduced to

$$\sum_{i=1}^{v} p_i = Q \neq 1 \quad (49)$$

where p_i is the true probability which can not sum to one. The necessity to introduce this nonextensive or nonadditive probability is first noted by economists [1]. This probability in equation (49) was referred to by Rényi as *incomplete probability distribution* [21] because the values of the random variable of this distribution do not constitute a complete (exhaustive) ensemble (*i.e.* $v < w$).

It should be noticed that Q is a constant depending on the studied system. Now in order to apply conventional probability theory, equation (49) has to be renormalized to get a calculable "probability" related to p_i as well as to Q representing the nature of the system.

Our proposal [2] is to write

$$\sum_{i=1}^{v} \frac{p_i}{Q} = \sum_{i=1}^{v} p_i^q. \tag{50}$$

which means

$$\sum_{i=1}^{v} p_i^q = 1, (q \in [0, \infty]) \tag{51}$$

and, for internal energy,

$$U = \sum_{i=1}^{v} p_i^q e_i. \tag{52}$$

Since p_i is the true probability and satisfies $0 \leq p_i < 1$, we have to set $0 < q < \infty$. $q = 0$ should be avoided because it leads to $p_i = 0$ for all states. We see from equation (50) that $Q = 1$, $Q < 1$ and $Q > 1$ means $q = 1$, $q < 1$ and $q > 1$, respectively. q is directly related to Q (*e.g.*, for microcanonical distribution, $q = \frac{\ln v}{\ln v - \ln Q}$) and, in this way, to the unknown correlations or information. This may help to understand empirical q values for nonextensive systems.

6 Nonextensive statistics

In this section, we present a method based on the incomplete normalization equation (51) in order to get TDF.

6.1 Nonextensive information and entropy

On the basis of the hypothesis of the nonextensivity (nonadditivity) of entropy or other quantities, we proposed using a generalized logarithm function as a generalized Hartley formula for the information measure $I(N)$ required to specify one element in a system containing N elements:

$$I(N) = \frac{N^g - 1}{g} \tag{53}$$

where g is a real number. This means the following nonadditivity:

$$I(N_1 \times N_2) = I(N_1) + I(N_2) + gI(N_1) \times I(N_2) \tag{54}$$

where $I(N_1 \times N_2)$ is the information needed to specify simultaneously 2 elements, each being in a subsystem 1 or 2. We see that the parameter g is a measure of nonextensivity. If $g \to 0$, $I(N) \to \ln N$ and the information becomes extensive (additive). Equations (52, 53) or (54), plus the other axioms used by Shannon [2,22], lead to a nonextensive entropy

$$S = -k \sum_{i=1}^{v} p_i^q \frac{p_i^g - 1}{g} = -k \frac{\sum_{i=1}^{v} p_i^{q+g} - 1}{g}. \tag{55}$$

We should ask that the above entropy recover the Gibbs-Shannon one $S = -k \sum_{i=1}^{w} p_i \ln p_i$ for $q = 1$. This constraint on nonextensive entropy is logical because $q = 1$ or $Q = 1$ implies a complete knowledge about the studied system and short range interactions or correlations. In this case, we do not have any reason for holding the nonextensivity. So $g = 0$ when $q = 1$ and g should be monotonic function of q, which ensures the monotonic q-dependence of entropy.

6.2 Generalized distribution function

For maximum entropy, we write the following auxiliary function:

$$A = \frac{1 - \sum_{i=1}^{v} p_i^{q+g(q)}}{g(q)} + \alpha\gamma \sum_{i}^{v} p_i^q - \alpha\beta' \sum_{i}^{v} p_i^q e_i. \tag{56}$$

Let $\frac{\mathrm{d}A}{\mathrm{d}p_i} = 0$, we obtain

$$p_i = \frac{\left[1 - \frac{\beta'}{\gamma} e_i\right]_{\dagger}^{\frac{1}{g(q)}}}{Z} \tag{57}$$

with

$$Z^q = \sum_{i}^{v} \left[1 - \frac{\beta'}{\gamma} e_i\right]_{\dagger}^{\frac{q}{g(q)}}. \tag{58}$$

Now it should be asked that the distribution equation (57) become the BGS exponential distribution for $q = 1$ or $g(q) = 0$, which means that equation (57) should be a generalized exponential function, that is

$$\frac{Z^g p_i^g(e_i) - 1}{g} = -\beta' e_i. \tag{59}$$

This straightforwardly leads to $\gamma = 1/g(q)$ and

$$p_i = \frac{[1 - g(q)\beta' e_i]_{\dagger}^{\frac{1}{g(q)}}}{Z} \tag{60}$$

From equation (60), it can be shown that

$$\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2} = -[g(q) + q] p_i^{g(q)+q-2}. \tag{61}$$

If we want that the distribution equation (57) be a maximum entropy (or minimum energy [23]) distribution, we should put

$$g(q) + q > 0 \tag{62}$$

which ensures $\frac{\mathrm{d}^2 A}{\mathrm{d}p_i^2} < 0$ for a maximum entropy with equation (56). This means that the curve of $g(q)$ is situated above the straight line of $g(q) = -q$. If we impose the monotonicity of S and $g(q)$ with respect to q, we also have

$$\frac{\mathrm{d}g(q)}{\mathrm{d}q} < 0. \tag{63}$$

Equation (60) is the generalized nonextensive distribution function. Comparing it to equation (1), we see that TDF corresponds to $g(q) = 1 - q$. This choice is the simplest one that satisfies the two conditions equations (62, 63) and also yields the simplest nonextensive entropy:

$$S = -k\sum_{i=1}^{v} p_i^q \frac{p_i^{1-q}-1}{1-q} = -k\sum_{i=1}^{v}\frac{p_i - p_i^q}{1-q}$$
$$= k\frac{1-\sum_{i=1}^{v} p_i}{1-q}\,. \tag{64}$$

With the simplest generalized distribution function

$$p_i = \frac{[1-(1-q)\beta' e_i]_\dagger^{\frac{1}{1-q}}}{Z}\,. \tag{65}$$

Which is the TDF in IS version. It should be remembered that a different forms of $g(q)$ may lead to different distributions.

Note that the distribution equation (65) is in the same form as the TDF in Curado-Tsallis version but, due to the incomplete normalization, has a different partition function given by

$$Z^q = \sum_i^v [1-(1-q)\beta' e_i]_\dagger^{\frac{q}{1-q}}. \tag{66}$$

The consequences of this change will be discussed in the following sections.

We would like to mentioned here that Kaniadakis [24,25] proposed a new generalization of BGS (κ-deformed statistics) on the basis of nonlinear kinetics in low density gas systems with normalized distribution function f. It is argued that the q-variance of the q-exponential function of TDF suggests to write $f = p_i^q$ and so equations (51, 52). This work gives equation (65) as single particle distribution from the nonlinear kinetics theory. This result is in accordance with the conclusion of reference [18] that TDF is an exact distribution for both many-body system (nonextensive) and correlated single body according to equation (18) prescribed by thermodynamic equilibrium [17,18]. In this sense, the hypothesis of *low density gas* is no longer necessary, and the κ-deformed statistics can be regarded as a valid theory for any nonextensive gas system in equilibrium having Tsallis entropy.

6.3 Generalized distribution and nonextensivity

We consider again the total system C composed of two subsystems A et B in interaction. By $p_{ij}(C)$ we denote the probability that C is at the product state ij while A is at the state i with a probability $p_i(A)$ and B at j with a conditional probability $p_{ij}(B \mid A)$. We can write

$$p_{ij}(C) = p_i(A)p_{ij}(B \mid A), \tag{67}$$

or

$$p_{ij}^q(C) = p_i^q(A)p_{ij}^q(B \mid A), \tag{68}$$

which means

$$\frac{[1-g(q)\beta' e_{ij}(C)]_\dagger^{\frac{q}{g(q)}}}{Z^q(C)} = \frac{[1-g(q)\beta' e_i(A)]_\dagger^{\frac{q}{g(q)}}}{Z^q(A)}\,\frac{[1-g(q)\beta' e_{ij}(B)]_\dagger^{\frac{q}{g(q)}}}{Z^q(B \mid A)} \tag{69}$$

or

$$e_{ij}(C) = e_i(A) + e_{ij}(B) - g(q)\beta' e_i(A)e_{ij}(B) \tag{70}$$

and, from equation (52),

$$U(C) = U(A) + U(B) - g(q)\beta' U(A)U(B). \tag{71}$$

Since S is an observable as the others, equation (71) must hold for S as well. Indeed, if we put the multiplication law equation (67) into the entropy equation (55), we get

$$\begin{aligned} S(C) &= -k\frac{\sum_{ij} p_{ij}^{q+g(q)}(C) - 1}{g(q)} \\ &= -k\frac{\sum_{ij} p_i^{q+g(q)}(A)p_{ij}^{q+g(q)}(B \mid A) - 1}{g(q)} \\ &= -k\frac{\sum_i p_i^{q+g(q)}(A) - 1}{g(q)} - k\frac{\sum_j p_{ij}^{q+g(q)}(B \mid A) - 1}{g(q)} \\ &\quad - \frac{g(q)}{k}k^2 \\ &\quad \times \frac{\sum_i p_i^{q+g(q)}(A) - 1}{g(q)}\,\frac{\sum_j p_{ij}^{q+g(q)}(B \mid A) - 1}{g(q)} \\ &= S(A) + S_i(B \mid A) - \frac{g(q)}{k}S(A)S_i(B \mid A). \end{aligned} \tag{72}$$

with

$$S_i(B \mid A) = -k\frac{\sum_j p_{ij}^{q+g(q)}(B \mid A) - 1}{g}. \tag{73}$$

Following the idea of Abe *et al.* [26], we can define a conditional entropy of B as follows

$$S(B \mid A) = \sum_i p_i^q(A)S_i(B \mid A). \tag{74}$$

From equation (72), we simply obtain

$$\begin{aligned} S(B \mid A) &= \sum_i p_i^q(A)\frac{S(C) - S(A)}{1 - \frac{g(q)}{k}S(A)} \\ &= \frac{S(C) - S(A)}{1 - \frac{g(q)}{k}S(A)} \end{aligned} \tag{75}$$

or

$$S(C) = S(A) + S(B \mid A) - \frac{g(q)}{k} S(A) S(B \mid A), \quad (76)$$

which has exactly the same form as equation (71). It should be noticed that this equation means that $S(B \mid A) = S_i(B \mid A)$, *i.e.* the conditional entropy of B associated with a microstate i of A is in fact independent of i. It is a little surprising to see this property with two correlated subsystems. But if we look back to the origin, we find that this is simply the consequence of the postulate equation (54) which supposes a nonextensive term in the form of product of two sub-informations. This is naturally a special choice of the nonextensivity described by the generalized Hartley formula. If we choose a different nonextensive information measure, the things will be different. Recently, the proposition of an entropy pseudoadditivity [17] prescribed by thermal equilibrium shed light on this problem. We understand that, supposed the generalized Hartley formula (or Tsallis entropy), equation (54) and so equation (22) are prescribed by the existence of thermal equilibrium. In other words, without these relations, a composite system with correlated (not independent!) subsystems can not have stable equilibrium state. So any exact discussion about equilibrium systems must conform with the factorization of compound probability equation (18). As a consequence, the idea of additive energy with neglected correlations [15,16] or non interacting systems [11], incompatible with the spirit of nonextensive statistics, becomes unnecessary and should be rejected. The definition of temperature has to be revisited on the basis of nonextensive energy satisfying equation (18). It is what we are doing in the following section.

7 Thermodynamic relations

To give statistical interpretation of thermodynamics, a well defined temperature related to stable thermodynamic equilibrium (maximum entropy or minimum energy) is needed. In this section, we will present briefly some consequences of IS. All the discussions are based on equation (18) and compatible entropy and energy pseudoadditivities.

7.1 Zeroth law and generalized temperature

First, what is β' in the distribution equation (60)? In BGS, $\beta' = \beta = \frac{1}{kT}$ is the inverse temperature and the first law of thermodynamics can be written as

$$\mathrm{d}U = \mathrm{d}Q + W \quad (77)$$

or

$$\mathrm{d}U = T\mathrm{d}S + Y\mathrm{d}X \quad (78)$$

where W is the work done by Y, a generalized exterior force (*e.g.* pressure $-P$), X the correspondent displacement (*e.g.* volume V), and $\mathrm{d}S = \frac{\mathrm{d}Q}{T}$ the thermodynamic definition of entropy. When X remains constant, we have $\mathrm{d}U = \mathrm{d}Q$ and

$$\frac{\mathrm{d}S}{\mathrm{d}U} = \frac{1}{T} = k\beta. \quad (79)$$

On the other hand, within IS, from equations (55, 60), we obtain

$$S = -k\frac{Z^{-g(q)} - 1}{g(q)} + k\beta' Z^{-g(q)} U. \quad (80)$$

or

$$S' = -k\frac{1 - Z^{1-q}}{1-q} + k\beta' U \quad (81)$$

with $S' = Z^{g(q)} S = Z^{1-q} S$. This leads to

$$\frac{\mathrm{d}S'}{\mathrm{d}U} = k\beta'. \quad (82)$$

Now we are showing that β' still measures thermal equilibrium at *maximal entropy or minimal energy*. Let us take the nonextensivity relation equation (71) and calculate a small variance of energy $U(C)$:

$$\delta U(C) = [1 - g(q)\beta' U(B)]\delta U(A) + [1 - g(q)\beta' U(A)]\delta U(B). \quad (83)$$

At equilibrium or *energy minimum*, $\delta U(C) = 0$, we obtain

$$[1 - g(q)\beta' U(B)]\delta U(A) = - [1 - g(q)\beta' U(A)]\delta U(B). \quad (84)$$

Putting this equation into the *entropy maximum relation* equation (24) [in which $(1-q)$ should be replaced by $(q-1)$ due to the IS version of entropy nonextensivity equation (76)], we get

$$\frac{1 - g(q)\beta' U(A)}{1 - \frac{g(q)}{k} S(A)} \frac{\partial S(A)}{\partial U(A)} = \frac{1 - g(q)\beta' U(B)}{1 - \frac{g(q)}{k} S(B)} \frac{\partial S(B)}{\partial U(B)} \cdot \quad (85)$$

With the help of equation (80), we can establish

$$Z^{g(q)}(A)\frac{\partial S(A)}{\partial U(A)} = Z^{g(q)}(B)\frac{\partial S(B)}{\partial U(B)} \quad (86)$$

or

$$\frac{\partial S'(A)}{\partial U(A)} = \frac{\partial S'(B)}{\partial U(B)} \quad (87)$$

which means

$$\beta'(A) = \beta'(B). \quad (88)$$

This result can also be obtained in another way. Multiplying equation (76) by $Z^{1-q}(C)$ and considering $Z(C) = Z(A)Z(B)$, we obtain

$$S'(C) = Z^{1-q}(B) S'(A) + Z^{1-q}(A) S'(B) - \frac{g(q)}{k} S'(A) S'(B). \quad (89)$$

On the other hand, from equation (80), it is straightforwardly to verify that, for maximum entropy at equilibrium, $\delta Z(C) = 0$ and $\delta S'(C) = 0$. This leads equation (89) to

$$\left[Z^{1-q}(B) - \frac{1-q}{k}S'(B)\right]\delta S'(A) = -\left[Z^{1-q}(A) - \frac{1-q}{k}S'(A)\right]\delta S'(B). \quad (90)$$

Now with the help of equations (81, 84), we obtain equations (87, 88).

So β' remains the meter of stable equilibrium state. We can define a generalized temperature

$$T' = \frac{1}{k\beta'} \quad (91)$$

where $T' = T$ if $q = 1$ or $g(q) = 0$.

7.2 Some other relations

Considering equation (82) and the energy conservation law equation (77), we see that $\mathrm{d}S'$ should be the measure of heat transfer. We have to write

$$\mathrm{d}Q = \frac{\mathrm{d}S'}{k\beta'}, \quad (92)$$

or

$$\mathrm{d}Q = T'\mathrm{d}S'. \quad (93)$$

Now the first law of thermodynamics equation (78) should be written as follows

$$\mathrm{d}U = T'\mathrm{d}S' + Y\mathrm{d}X. \quad (94)$$

The free-energy F have to be defined as

$$\mathrm{d}F = -S'\mathrm{d}T' + Y\mathrm{d}X \quad (95)$$

or

$$F = U - T'S', \quad (96)$$

which leads to, with the help of equation (81):

$$F = -kT'\frac{Z^{1-q} - 1}{1-q}\cdot \quad (97)$$

Considering $Z(C) = Z(A)Z(B)$, we easily obtain:

$$F(C) = F(A) + F(B) - \frac{g(q)}{kT'}F(A)F(B). \quad (98)$$

We also have, for the heat capacity

$$C_X = \frac{\mathrm{d}Q}{\mathrm{d}T'} = T'\left\{\frac{\partial S'}{\partial T'}\right\}_X = -T'\left\{\frac{\partial^2 F}{\partial T'^2}\right\}_X \quad (99)$$

and for the generalized force

$$Y = \left\{\frac{\partial F}{\partial X}\right\}_{T'}\cdot \quad (100)$$

8 The fundamental problems revisited within incomplete statistics

What about the problems discussed in Section 3 if we consider IS? Let us examine them one by one.

1. The first problem of the incompatibility between normalization and expectation value does not exist any more because the incomplete normalization equation (51) is compatible with the expectation value equation (52).
2. The second problem was that the expectation value equation (3) was not linear in the distributions shown by equation (16). This problem can be avoided within IS thanks to equations (51, 52):

$$\begin{aligned}\overline{x}[\lambda p(1)+(1-\lambda)p(2)] &= \frac{\sum_i[\lambda p_i^q(1)+(1-\lambda)p_i^q(2)]x_i}{\sum_i[\lambda p_i^q(1)+(1-\lambda)p_i^q(2)]}\\ &= \lambda\sum_i p_i^q(1)x_i\\ &\quad + (1-\lambda)\sum_i p_i^q(2)x_i\\ &= \lambda\overline{x}[p(1)] + (1-\lambda)\overline{x}[p(2)].\end{aligned} \quad (101)$$

3. The problem of the absence of analytic correlation in energy and other observable quantities can be resolved by equations (70, 71), which guarantee a nonextensive statistical theory with mathematical rigor.
4. The problem of the zeroth law of thermodynamics was resolved in the above section.
5. The fifth problem discussed in Section 3 is due to the peculiar equality equation (30) which in turn is due to the average defined in equation (3). In IS, this problem does not exist any more thanks to the incomplete normalization and the concomitant expectation value equations (51, 52).
6. In IS, the $U - Z$ relation can be found from equations (52, 65):

$$\begin{aligned}U &= \sum_i p_i^q e_i\\ &= \frac{1}{Z^q}\sum_i e_i[1-(1-q)\beta' e_i]_\dagger^{\frac{q}{1-q}}\\ &= -\frac{1}{Z^q}\frac{\partial}{\partial\beta'}\sum_i[1-(1-q)\beta' e_i]_\dagger^{\frac{1}{1-q}}\\ &= -\frac{1}{Z^q}\frac{\partial\overline{Z}}{\partial\beta'}\end{aligned} \quad (102)$$

where

$$\overline{Z} = \sum_i[1-(1-q)\beta' e_i]_\dagger^{\frac{1}{1-q}}. \quad (103)$$

9 Energy invariance of the distribution

A crucial problem of the nonextensive distribution equation (65) is that it is not invariant with respect to uniform translation of energy spectra e_i. If we replace e_i by $e_i + C$ where C is constant, equation (65) becomes:

$$p_i = \frac{[1-(1-q)\beta'(e_i+C)]_\dagger^{\frac{1}{1-q}}}{Z} \tag{104}$$

with

$$Z^q = \sum_i^v [1-(1-q)\beta'(e_i+C)]_\dagger^{\frac{q}{1-q}} \tag{105}$$

which is not same as equation (65), excepted that $q = 1$. This problem worries enormously scientists [4]. It is known that thermostatistics takes into account only energies relative to thermodynamic movements and that the choice of energy-zero is never a problem in BGS because the theory has exponential distribution and is invariant with uniform energy translation. But with TDF, equation (104) implies that the properties of a gas may depend on the translation speed or on the location of the container. Although in practice we can always choose the usual energy-zero as for BGS, this peculiar theoretical property of TDF is somewhat unusual and disturbing. Avoiding this puzzling variance of distribution has been one of the motivations of the third version of Tsallis theory. Now the following questions arise: Why do we have to fix only one choice of zero-energy in TDF to avoid the container-dependence of the statistics? Why does the nonextensivity lead to this theoretical property? Is it really something to be avoided?

As is well known, BGS is an extensive theory which holds only for systems with weak or short range interactions. In addition, the invariance of BGS is based on the classical mechanics which leaves the interaction potential completely arbitrary. But TDF is a theory for solving problems with complex interaction or correlations we often ignore. So we are not obliged or it is not advisable to identify the variance of TDF to that of BGS. The energy translation invariance is not an universal characteristic of physical theories. As a matter of fact, this property of arbitrary potential energy disappears even in classical mechanics and BGS if we consider the relativistic effect. We can not add constant into the energy E of a system because E is related to its total mass M by

$$E = Mc^2 \tag{106}$$

where c is the light speed. E is not arbitrary because M can not be changed arbitrarily.

If we accept the the variance of TDF, we have to choose a definite zero potential energy. Let us suppose a non negligible interaction or correlation between all the elements of mass m_i and energy e_i $(i = 1, 2, 3, ...)$ of a system. Let V be the total interaction energy. M can be given by:

$$M = \sum_i m_i + V/c^2 \tag{107}$$

or

$$Mc^2 = \sum_i m_i c^2 + V = \sum_i e_i + V \tag{108}$$

with $e_i = m_i c^2$ for i^{th} element. It is obvious that neither M, nor e_i and V can be changed. If not, the variance of the theory would be perturbed [29]. According to this discussion, a possible choice of zero potential energy corresponds to the following case

$$M = \sum_i m_i \tag{109}$$

or $V = 0$. This condition may correspond in some cases to infinite distance between the elements of the system (*e.g.* an atom when we consider the internal energy between the electrons and the nucleus), and in other cases, to special positions of this elements (*e.g.* equilibrium position of the atoms in crystal lattice if we are interested in their vibration). This is in fact just what we usually choose with BGS.

It is worth emphasizing that some of the problems discussed in Section 3 seem to be related to the distribution functions invariant through energy translation. Because, if we apply the maximization method of Martinez *et al.* [13], *i.e.* to introduce $\sum_i^w p_i^q U$ as the invariance constraint into the auxiliary function equation (56), we obtain:

$$p_i = \frac{[1-(1-q)\beta'(e_i-U)]_\dagger^{\frac{1}{1-q}}}{Z} \tag{110}$$

with

$$Z^q = \sum_i^v [1-(1-q)\beta'(e_i-U)]_\dagger^{\frac{q}{1-q}}, \tag{111}$$

which is invariant through energy translation and different from equation (1) only by the partition function. We easily find that the problems 3 to 6 take place again with questionable equalities similar to equations (19, 24) and (30) which make it difficult to establish the zeroth law without any approximation and lead to other puzzling equalities like equations (37, 40) and (46). Does the distribution invariance inevitably lead to the theoretical peculiarities? This seems an interesting topic which is beyond the range of the present work.

10 Conclusion

We have shown some observations about the actual nonextensive statistical theory. The problems discussed reveal that, with the conventional normalization equation (4) and the expectation value equation (3), the generalized exponential distribution, though very successful in many applications, cannot be obtained with convincing theoretical approach and so the nonextensive statistics shows peculiar properties which seem difficult to be avoided. We have shown that it was possible to overcome these difficulties if we introduced the concept of incomplete information with suitable normalization and expectation. This approach allows to establish TDF in a consistent way with only a different partition function. New nonextensive thermodynamic relations were deduced on the basis of generalized definitions of heat and temperature. It is argued that the energy invariance should not be considered as a necessary property of TDF. The connection between the nonextensivity and the energy shift dependence of TDF remains to be understood.

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